PROPERTIES OF PORTFOLIO SHARPE RATIO AND THEIR APPLICATIONS

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For a number of years I have been involved as a part of RINA Systems in developing portfolio analysis software. Once, during a conversation with Mr. Jack Schwager (Schwager [1996]) he shared with me his observation that he had not seen Sharpe ratio of a portfolio to be smaller than the smallest Sharpe Ratio of individual components. The article describes properties of portfolio Sharpe Ratio as a function of Sharpe Ratio of the portfolio components. It shows the conditions when Mr. Schwager's observation is correct. This property is then used to build a portfolio with the highest Sharpe Ratio possible from a list of components as well as a portfolio with the highest net profit and Sharpe Ratio no less than a given value. These results are used in the software for portfolio analysis and design.

Definitions.

Let $X_i = \{x_{i1},...,x_{im}\}, i = \overline{1,n}$ be n independent random variables with the values $x_{i1},...,x_{im}$, $m > 1$.
Let us assume that there are at least two different values for each random variable. Let μ_i, V_i and σ_i be average, variance and standard deviation of X_i correspondingly.
Let $X_s = \{x_{s1},...,x_{sm}\}$ be a random variable that takes the values equal to the sum of values for all X_i, $i = \overline{1,n}$, correspondingly. So,

$$x_{s_j} = \sum_{i=1}^{n} x_{ij}\ ,\ \ j = \overline{1,m} \qquad (1)$$

Let μ_s, V_s and σ_s be average, variance and standard deviation of X_s. Then it is possible to state the following.

Lemma.

For all i such that $\mu_i \geq 0$

$$\frac{\mu_s}{\sigma_s} \geq \min_i \left\{ \frac{\mu_i}{\sigma_i} \right\}, i = \overline{1,n}. \qquad (2)$$

Proof.

Let

$$\min_{1 \leq i \leq n} \left\{ \frac{\mu_i}{\sigma_i} \right\} = \frac{\mu_k}{\sigma_k}. \qquad (3)$$

This means that

$$\frac{\mu_k}{\sigma_k} \le \frac{\mu_i}{\sigma_i} \quad \text{for all } i = \overline{1,n},\ i \ne k. \tag{3a}$$

Because σ_i >0 we can rewrite (3a) as $\mu_k \sigma_i \le \mu_i \sigma_k$ for all $i = \overline{1,n}$, $i \ne k$. And because of the assumption that $\mu_i \ge 0$ we have

$$\mu_k^2 \sigma_k^2 \le \mu_i^2 \sigma_k^2 \quad \text{for all } i = \overline{1,n},\ i \ne k. \tag{4}$$

Or

$$\sum_{i=1, i \ne k}^{n} \mu_k^2 \sigma_k^2 \le \sum_{i=1, i \ne k}^{n} \mu_i^2 \sigma_k^2 . \tag{5}$$

Let us assume the opposite, namely that (2) is not correct. That means that

$$\frac{\mu_s}{\sigma_s} < \frac{\mu_k}{\sigma_k}. \tag{6}$$

From (1)

$$\mu_S = \sum_{i=1}^{n} \mu_i . \tag{7}$$

It is also known that for independent variables $V_s = \sum_{i=1}^{n} V_i$, $\sigma_s^2 = \sum_{i=1}^{n} \sigma_i^2$ and

$$\sigma_s = \sqrt{\sum_{i=1}^{n} \sigma_i^2} . \tag{8}$$

From (6) – (8) we get

$$\frac{\sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} < \frac{\mu_k}{\sigma_k}. \tag{9}$$

Or (because of the assumption $\mu_i \ge 0$ for all $i = \overline{1,n}$)

$$\left(\sum_{i=1}^{n}\mu_i\right)^2\sigma_k^2 < \mu_k^2\sum_{i=1}^{n}\sigma_i^2 \; . \qquad (10)$$

Because $\sum_{i=1}^{n}\mu_i^2 \le \left(\sum_{i=1}^{n}\mu_i\right)^2$ we can rewrite (10) as

$$\sigma_k^2\sum_{i=1}^{n}\mu_i^2 < \mu_k^2\sum_{i=1}^{n}\sigma_i^2 \; .. \qquad (11)$$

Eliminating $\mu_k^2\sigma_k^2$ from both parts of (11) we get

$$\sum_{i=1, i\ne k}^{n}\mu_i^2\sigma_k^2 < \sum_{i=1, i\ne k}^{n}\mu_k^2\sigma_i^2 \; . \qquad (12)$$

(12) is a contradiction to (5). That proves the lemma.

Sharpe Ratio Properties.

Sharpe Ratio has been introduced in Sharpe [1966]. In the sense of the article by Sharpe [1994] the Sharpe Ratio considered below is the Ex Post Sharpe Ratio.

Let now X_i be a vector of monthly returns for the i-th investment, X_s is a vector of returns for the portfolio that includes all the investments, for $i=\overline{1,n}$. Let I be the benchmark investment (e.g., interest rate) assumed for the period in consideration, SR_i is the Sharpe Ratio of returns for the i-th investment and SR_s is the Sharpe Ratio of the portfolio monthly returns for the portfolio that includes all n investments, $i=\overline{1,n}$.

Theorem.

For any portfolio of n investments with the average monthly return no less than the benchmark investment (e.g., interest rate) I

$$SR_s \ge \min_i\{SR_i\}, \; i=\overline{1,n} \; . \qquad (13)$$

Proof.

By definition,

$$SR_i = \frac{\mu_i - I}{\sigma_i}, \; SR_s = \frac{\mu_s - I}{\sigma_s} \; .$$

Let $\mu_i^{'} = \mu_i - I$, $\mu_s^{'} = \mu_s - I$. Because of the theorem assumption $\mu_i^{'} \geq 0$ and $\mu_s^{'} \geq 0$. Then from the lemma we have

$$\frac{\mu_s^{'}}{\sigma_s} \geq \min_i \left\{ \frac{\mu_i^{'}}{\sigma_i} \right\}, i = \overline{1, n}$$

and that proves the theorem.

Examples and Algorithms.

In the examples and algorithms below we assume for simplicity $I = 0$. This does not limit the results because if $I > 0$ it is always possible to make a substitution $\mu_i^{'} = \mu_i - I$, $\mu_s^{'} = \mu_s - I$ for corresponding I and use $\mu^{'}$ instead of μ in the statements that follow.
We also do not consider correlation of one strategy return with another.

1. Let $\mu_1 = -5$, $\sigma_1 = 5$, $\mu_2 = -1$, $\sigma_2 = 1$. Then

$$\frac{\mu_s}{\sigma_s} = \frac{-6}{\sqrt{26}} < -1 = \min\left\{ \frac{-5}{5}, \frac{-1}{1} \right\}.$$

If the average returns are negative Sharpe Ratio for the portfolio could be smaller than the minimum of all.

2. Let $\mu_1 = 5$, $\sigma_1 = 3$, $\mu_2 = 6$, $\sigma_2 = 4$. Then

$$\frac{\mu_s}{\sigma_s} = \frac{11}{5} > \max\left\{ \frac{5}{3}, \frac{6}{4} \right\}.$$

Sharpe Ratio of a portfolio (of more than one investment) could be larger than the highest of all individual Sharpe Ratios.

3. Adding an investment to a portfolio may increase or decrease the portfolio Sharpe Ratio. It increases if

$$\frac{\mu_s}{\sigma_s} < \frac{\mu_s + \mu_i}{\sqrt{\sigma_s^2 + \sigma_i^2}} \qquad (14a)$$

and decreases if

$$\frac{\mu_s}{\sigma_s} > \frac{\mu_s + \mu_i}{\sqrt{\sigma_s^2 + \sigma_i^2}}, \qquad (14b)$$

where μ_i and σ_i are the average and standard deviation of monthly returns for the investment to be added.
For example, let μ_s = 1, σ_s = 1 for a portfolio with Sharpe Ratio of 1. If an investment to be added to the portfolio has Sharpe Ratio of 0.9 with μ_i = 0.9, σ_i = 1 then adding this investment to the portfolio increases Sharpe Ratio because $\frac{1+0.9}{\sqrt{1^2+1^2}} = \frac{1.9}{\sqrt{2}}$ >1. If, however, μ_i = 9, σ_i = 10 with the same Sharpe Ratio of 0.9 then adding it to the portfolio decreases Sharpe Ratio because $\frac{1+9}{\sqrt{1^2+10^2}} = \frac{10}{\sqrt{101}}$ <1.

This example shows that having a high Sharpe Ratio for an investment is not sufficient to add this market to a portfolio for the purpose of increasing the portfolio Sharpe Ratio.
It is also easy to notice that combining investments that have the same average returns and standard deviation increases Sharpe Ratio because

$\frac{\mu_s + n\mu_s}{\sqrt{\sigma_s^2 + n^2\sigma_s^2}} = \frac{n+1}{\sqrt{n^2+1}}\frac{\mu_s}{\sigma_s} > \frac{\mu_s}{\sigma_s}$, where n is the number of investments added for all n>1.

Let us now consider an algorithm for creating a portfolio with maximum Sharpe Ratio from the investments available.

Algorithm 1.
Let us assume that there are n investments to be considered with average monthly returns μ_i and standard deviations σ_i for $i = \overline{1,n}$ in a list of investments L.
Step 1. Let $\mu_s = 0, \sigma_s = 0$.
Step 2. Find i^* such that the expression $\frac{\mu_s + \mu_i}{\sqrt{\sigma_s^2 + \sigma_i^2}}$ reaches maximum at $i = i^*$ or

$$i^* = \arg\max_{i \in L}\left(\frac{\mu_s + \mu_i}{\sqrt{\sigma_s^2 + \sigma_i^2}}\right) \qquad (15)$$

Step 3. If $\frac{\mu_s + \mu_{i^*}}{\sqrt{\sigma_s^2 + \sigma_{i^*}^2}} < \frac{\mu_s}{\sigma_s}$ then end of the algorithm.

If $\frac{\mu_s + \mu_{i^*}}{\sqrt{\sigma_s^2 + \sigma_{i^*}^2}} > \frac{\mu_s}{\sigma_s}$ then include the market i^* in the portfolio, recalculate μ_s and σ_s, and delete the market i^* from the list L.

If the list L is not empty go to Step 2. If the list L is empty then end the algorithm.

It is clear that the maximum number of selections is limited by the number of investments squared because

$$\sum_{i=1}^{n}(n+i-1) = \frac{1}{2}(3n^2 - n) = O(n^2)$$

In addition, this algorithm allows maximization of portfolio Sharpe Ratio without calculating on each iteration monthly portfolio returns X_s.

References

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